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Stationary Scattering Theory for a Charged Particles Transport Problem

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We consider a nonautonomous transport problem, the modelization of the charge exchange dynamics in a monoatomic ionized gas, and apply scattering theory to its dynamics. The free dynamics corresponds to the evolution of the total distribution of particles (neutral plus ionized particles) and the perturbed dynamics corresponds to the evolution of the neutral particles, which is the solution of a nonautonomous transport problem. The existence of the time-dependent wave operators was proved by the first author. In the present paper we follow Howland's formalism in constructing a stationary scattering theory for this nonautonomous transport problem by studying the evolution equation. We prove the existence of the wave operators and by using the smooth perturbation technique we obtain the similarity between perturbed and unperturbed operators.

KEY WORDS: Charged particle transport problem; stationary scattering theory; wave operators; scattering operator; smooth perturbation.

1. INTRODUCTION

The physical problem concerns the time evolution of the distributions of neutral and ionized atoms of equal atomic weights taking into account only interactions giving rise to a charge exchange; that is, in a collision between an ionized atom and a neutral one, the first becomes neutral and the latter becomes ionized. This kind of phenomena is important in mixtures, or plasmas, with low densities so to be allowed to neglect the ion–ion and neutral–neutral interactions. This phenomenon is explained by

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the charge exchange effect; A_1 and A_2 being atoms of the same mass, a particle of type A_1 which is ionized and a particle of type A_2 which is neutral react producing a neutral particle of type A_1 and a charged particle of type A_2. This may be represented by the formula

$$A_1^+ + A_2 \to A_1 + A_2^+$$

A more thorough description of the underlying physics can be found in [G-Z-S]. There it is explained why the charge exchange without change in the sum of the internal energies of the two colliding particles influences the dynamics of the plasma and of its single components. Moreover the sum of the kinetic energies of the two particles remains unchanged, so we can consider these collisions to be elastic. We assume also that the charge density vanishes everywhere, but we disregard the electron distribution. We denote by $f = f(x, v, t)$ the distribution of neutral particles and by $g = g(x, v, t)$ the distribution of ions; $x \in \mathbb{R}^3$ denotes the position of the particle in the configuration space, $v \in V \subset \mathbb{R}^3$ denotes the velocity of the particle, $t \in \mathbb{R}$ denotes the time. The balance equations are

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = P_f - R_f \tag{1.1}$$

$$\frac{\partial g}{\partial t} + v \cdot \nabla_x g = P_g - R_g \tag{1.2}$$

where P_f, P_g, R_f, R_g denote the production and removal rates for neutral and ionized particles. For sake of simplicity, but allowing for preservations of mass, momentum, and energy during the single collision, under the assumption of equal mass for ions and neutral atoms, we put

$$P_f(x, v, t) = \sigma g(x, v, t) \int_V f(x, v', t)\, dv' = R_g(x, v, t) \tag{1.3}$$

$$P_g(x, v, t) = \sigma f(x, v, t) \int_V g(x, v', t)\, dv' = R_f(x, v, t) \tag{1.4}$$

The removal rate R_f of particle of type (f) is modelled with a total cross-section $\sigma(x, t) = \sigma$ times the total number of (g) particles present in x at time t, and it is proportional to the density of (f) particles. Clearly we have to introduce the dimensional factor σ that in the sequel will be put equal to one. During the collision between two particles of different types, the charged one yields its charge to the other (the neutral one) remaining left neutral. The neutral one obviously becomes charged. The velocities of the particles are assumed not to change. Therefore $P_f = R_g$ and $P_g = R_f$.

Here V is a measurable not necessarily bounded set of $\mathbb{R}^3$ whose measure $m(V)$ is finite: $m(V) \leqslant \infty$. The integro-differential equations are supplied with initial data for the two distributions. We can consider the corresponding abstract Cauchy problem in the space $L^1(\mathbb{R}^3 \times V)$, see [Bus2]. By putting $h = f + g$ we obtain the system

$$\frac{\partial h}{\partial t} + v \cdot \nabla_x h = 0 \tag{1.5}$$

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = -\sigma f \int_V h \, dv' + \sigma h \int_V f \, dv' \tag{1.6}$$

showing that the total density h evolves as a pure streaming for $t \in \mathbb{R}$. Its evolution depends on the initial distributions of ionized and neutral atoms. Equation (1.5) is called the *advection equation* and its corresponding Cauchy problem will be denoted by

$$\text{(AE)} \qquad \begin{cases} \dfrac{dh}{dt} = T_0 h := -v \cdot \nabla_x h \\ h(\cdot, 0) = h_0 \in X \end{cases}$$

Equation (1.6) for the neutral (or ion) density looks like a linear transport equation with time dependent absorption and production terms, which really are determined by the initial distributions.

Taking $\sigma = 1$, the Eq. (1.6) can be considered as a nonautonomous transport problem in the following abstract form

$$\text{(NTP)} \qquad \begin{cases} \dfrac{df}{dt} = [T_0 + A(t) + K(t)] \, f(t) \\ f(s) = \varphi \in X, \qquad s \in \mathbb{R} \end{cases}$$

where

$$[A(t) f](x, v) := -\left(\int_V h_0(x - tv', v') \, dv' \right) f(x, v) \qquad \text{for all} \quad (x, v) \in \mathbb{R}^3 \times V \tag{1.7}$$

and

$$[K(t) f](x, v) = h_0(x - tv, v) \int_V f(x, v') \, dv' \qquad \text{for all} \quad (x, v) \in \mathbb{R}^3 \times V \tag{1.8}$$

Given a nonautonomous Cauchy problem

$$\text{(NCP)} \qquad \begin{cases} \dfrac{df}{dt} = T(t)\, f(t) \\ f(s) = \varphi \in X, \qquad s \in \mathbb{R} \end{cases}$$

on a Banach space X with possibly unbounded operator $T(t)$, $t \in \mathbb{R}$, on X, the solutions of (NCP) can be expressed (under appropriate conditions, see e.g., [Gol], [Paz], [Tan]) by a family $\{U(t, s)\}_{(t, s) \in \mathbb{R}^2}$ in the space $\mathscr{L}(X)$ of bounded linear operators on X satisfying the following properties:

(P1) $(t, s) \mapsto U(t, s)$ from $\mathbb{R}^2$ into $\mathscr{L}(X)$ is strongly continuous;

(P2) $U(t, t) = I$, $\forall t \in \mathbb{R}$;

(P3) The Chapman–Kolmogorov equation holds, i.e.,

$$U(t, s)\, U(s, r) = U(t, r), \qquad \forall (t, s, r) \in \mathbb{R}^3$$

(P4) there is a constant $M \geqslant 1$ such that $\| U(t, s) \| \leqslant M$ for all $(t, s) \in \mathbb{R}^2$, $t \geqslant s$.

We call such family of operators a *bounded strongly continuous propagator* on X.

Let E be a space of X-valued functions on $\mathbb{R}$ which will be precised later. Given a bounded strongly continuous propagator $U(\cdot, \cdot)$, the formula

$$e^{t\mathscr{G}} \psi(\cdot) := U(\cdot, \cdot - t)\, \psi(\cdot - t) \qquad \text{for} \quad \psi \in E, \quad t \geqslant 0 \tag{1.9}$$

defines a strongly continuous bounded semigroup on E and is called *evolution semigroup* associated to the propagator $U(\cdot, \cdot)$. Recently a substantial literature drew attention to the properties of this semigroup. For generation and perturbation results we refer to [Nag], [Nei], [Nic], [Paq], [R-R-S1], [R-R-S2] and the references therein. Furthermore, the spectrum $\sigma(\mathscr{G})$ of the infinitesimal generator of $\{e^{t\mathscr{G}}\}_{t \geqslant 0}$ is responsible for the asymptotic behaviour of $u(t)$, via spectral mapping theorem. This is extensively discussed in [L-M], [ET-S1], [R-S2] and [Rau]. Originally this formalism was used by J. Howland [How] in the Hilbert space context and by D. Evans [Eva] in the Banach space context to indicate how the stationary theory of scattering can be applied to problems in the theory of transitions of a quantum mechanical system. Our purpose in this article will be the same for a charge exchange transport problem.

From a mathematical point of view the scattering theory which we are dealing with means investigating if the behaviour of a system which evolves according to a perturbed dynamics can be compared with that of a suitable unperturbed one. The scattering theory for the linear Boltzmann equation was initiated by J. Hejtmanek [Hej] and B. Simon [Sim]. This gave rise to a wealth of literatures on this topic (see [Voi], [Pro], [Um1], [Em1], [Um2], [Em2], [Ste], [A-E], [Mok], [E-P] and [Em3]). All these papers deal with the scattering theory for autonomous transport equation in L^1 spaces. Recently by assuming some smooth conditions on the initial distribution h_0, the first author treated the scattering theory for time dependent transport equation and showed in [Bus1] that (NTP) can be generated by a strongly continuous isometric propagator $U(\cdot, \cdot)$. In Section 2 we recall the main steps in constructing the propagator $U(\cdot, \cdot)$. Taking $U_0(t, s) := e^{(t-s)\, T_0}$, the existence of the wave operators

$$W_+(s)\, f := \lim_{t \to \infty} U_0(s, t)\, U(t, s)\, f \qquad \text{in } X$$

$$W_-(s)\, f := \lim_{t \to -\infty} U(s, t)\, U_0(t, s)\, f \qquad \text{in } X$$

is proven (see Theorem 2.1) by assuming that

$$\alpha_\pm := \int_{b_\pm} \sup_{x \in R^3} H(x, r)\, dr < \infty \tag{1.10}$$

where $b_+ \equiv (s, \infty)$, $b_- \equiv (-\infty, s)$ and $H(x, t) \equiv \int_V h_0(x - tv, v)\, dv$.

Once the wave operators $W_+(s)$, $W_-(s)$ and consequently the scattering operators $S(s) := W_+(s)\, W_-(s)$ are well-defined, Howland's programme [How] was to pass from the time-dependent case to stationary case by considering the new wave operators

$$\mathscr{W}_+(\mathscr{G}_0, \mathscr{G}) := s - \lim_{t \to +\infty} e^{-t\mathscr{G}_0} e^{t\mathscr{G}} \qquad \text{in } E \tag{1.11$_+$}$$

$$\mathscr{W}_-(\mathscr{G}, \mathscr{G}_0) := s - \lim_{t \to -\infty} e^{-t\mathscr{G}} e^{t\mathscr{G}_0} \qquad \text{in } E \tag{1.11$_-$}$$

where

$$e^{t\mathscr{G}_0}\psi(\cdot) := U_0(\cdot, \cdot - t)\, \psi(\cdot - t) \qquad \text{for} \quad \psi \in E \tag{1.12}$$

and show that the scattering operator

$$\mathscr{S} := \mathscr{W}_+(\mathscr{G}_0, \mathscr{G})\, \mathscr{W}_-(\mathscr{G}, \mathscr{G}_0) \tag{1.13}$$

is the multiplication by the operator $S(t)$.

Our main aim in the present work is to develop the above formalism for the theory which was introduced in [Bus1]. The Section 3 is devoted to the characterization of the infinitesimal generators of the evolution groups $e^{t\mathscr{G}_0}$ and $e^{t\mathscr{G}}$ corresponding to the Cauchy problems (AE) and (NTP). The relationship between these groups through Duhamel's formulae needs to be justified. In the Section 4, we will define the wave operators given in (1.11_+) and (1.11_-). The existence of these wave operators are guaranteed by imposing that $H(x, t)$ total cross-section of the initial total density has an uniformly bounded X-ray tranformation (see (4.3) and (4.7)). Similar conditions are used in [Bus1] for obtaining the wave operators $W(s)_+$ and $W(s)_-$.

The similarity between perturbed and unperturbed dynamics in the scattering theory passes through the existence of the four wave operators $\mathscr{W}_{\pm}(\mathscr{G}_0, \mathscr{G})$ and $\mathscr{W}_{\pm}(\mathscr{G}, \mathscr{G}_0)$. In the context of quantum scattering theory in Hilbert space, this is called the completeness of the wave operators and the intertwining identities imply the similarity between the the dynamics and their generators. T. Umeda [Um2], introduced this subject in the context of the Banach lattices and applied his technique to the linear transport equation in L^1 spaces. Later Mokhtar–Kharroubi [Mok] optimized the results of [Um2] in L^1 spaces. Their technique depends heavily upon the positivity preserving character of the free dynamics and perturbation operators. In the Section 5 we have used the smooth perturbation technique by maximal use of the isometric character of the free dynamics operator $e^{t\mathscr{G}_0}$.

2. STRONGLY CONTINUOUS PROPAGATOR AND SCATTERING OPERATOR FOR CHARGED PARTICLES TRANSPORT EQUATION

Put $X := L^1(\mathbb{R}^3 \times V)$, $Y := UCB(\mathbb{R}^3 \times V)$; they are the complex Banach spaces of (classes of) Lebesgue summable functions and of uniformly continuous bounded functions respectively, with the usual norms (see [B-B, p. 22]). They have the structure of Banach lattices with positive cones consisting of the nonnegative functions, which are called positive elements. Let $C_0^1(\mathbb{R}^3 \times V)$ be the space of continuously differentiable functions with compact support in $\mathbb{R}^3 \times V$. The operator defined in $C_0^1(\mathbb{R}^3 \times V)$ by $T_0 h := -v \cdot \nabla_x h$ has a closure in X denoted again by T_0, whose domain is $D(T_0) \subset C_0^1(\mathbb{R}^3 \times V)$. This closed operator generates a one parameter strongly continuous group $\{e^{tT_0}\}_{t \in \mathbb{R}}$ of bounded operators in X acting as

$$(e^{tT_0} f)(x, v) = f(x - vt, v) \tag{2.1}$$

having the property, for any $t \in \mathbb{R}$

$$\|e^{tT_0} f\|_X = \|f\|_X \tag{2.2}$$

and it preserves the cone of positive functions. We call T_0 the *streaming operator*. In the space Y we define the streaming operator $\tilde{T}_0$ on $D(\tilde{T}_0)$

$$D(\tilde{T}_0) = \{f \in Y : v \cdot \nabla_x f \in Y\}$$

$$(\tilde{T}_0 f)(x, v) = -v \cdot \nabla_x f(x, v)$$

It generates an isometric positive one parameter strongly continuous group of linear operators in Y acting as (2.1).

In either space X and Y the abstract Cauchy problem

$$\begin{cases} \dfrac{dh}{dt} = T_0 h \\ h(s) = h_0 \in D(T_0) \end{cases}$$

where T_0 stands for T_0 itself or for $\tilde{T}_0$, according to the space, has one and only one solution for any $t, s \in \mathbb{R}$, given by

$$h(t) = U_0(t, s)\, h(s) = e^{(t-s)\, T_0} h(s) = h_0(x - v(t-s), v)$$

We recall that by a solution in X, or in Y, we mean a function with values in X, or in Y such that $h(t) \in D(T_0)$, h is strongly continuously differentiable for $t \in \mathbb{R}$, the differential equation and the initial condition at the starting time s are satisfied. Some definitions are in order to present the analysis of evolution systems. Let $h_0 \in D(T_0) \cap Y^+$ (Y^+ being the positive cone in Y). For $t \in \mathbb{R}$, we define

$$H(x, t) := \int_V h_0(x - wt, w)\, dw \tag{2.3}$$

then for the operators $A(t)$ and $K(t)$ defined by (1.7) and (1.8) we have

$$(A(t)\, f)(x, v) := -H(x, t)\, f(x, t)$$

and

$$(K(t)\, f)(x, v) := h_0(x - vt, v) \int_V f(x, w)\, dw$$

The following statements are proved in [Bus1].

(i) $H(x, t) = \int_V e^{t\tilde{T}_0} h_0(x, w)\, dw$;

(ii) $\|A(t) f\|_X \leqslant m(V)\, \|h_0\|_Y\, \|f\|_X$ and $\|A(t)\| = \sup_x \int_V h_0(x - wt, w)\, dw$;

(iii) $A(t) f$ is continuously differentiable with respect to $t \in \mathbb{R}$ and

$$\frac{d}{dt} A(t)\, f(x, v) = -f(x, v) \int_V (e^{t\tilde{T}_0} \tilde{T}_0 h_0)(x, w)\, dw$$

(iv) $(K(t)\, f)(x, v) = (e^{t\tilde{T}} h_0)(x, v) \int_V f(x, w)\, dw$;

(v) $\|K(t) f\|_X \leqslant m(V)\, \|h_0\|_Y\, \|f\|_X$ and $\|K(t)\| = m(V) \sup_{(x, v)} h_0(x, v)$

(vi) $K(t) f$ is continuously differentiable with respect to $t \in \mathbb{R}$ and

$$\frac{d}{dt} K(t)\, f(x, v) = \int_V f(x, w)\, dw (e^{t\tilde{T}_0} \tilde{T}_0 h_0)(x, v)$$

We recall that $m(V)$ is the measure of V. Such results are enough (see [Bus1] and [Paz]) to prove that $T_0 + A(t)$ generates an evolution system on X, say $U_a(t, s)$, and $T(t) := T_0 + A(t) + K(t)$ generates another evolution system on X, say $U(t, s)$. It is also worth to recall that for any $t \geqslant s$ one has

$$\|U(t, s)\| = 1 \tag{2.4}$$

(see [Bus 1, Theorem 2.1]).

With these preliminaries it is possible to consider in the space $\mathscr{X} = X^2$ the Cauchy problem for the densities of neutral and ionized particles

$$\text{(PTP)} \qquad \begin{cases} \dfrac{\partial f}{\partial t} = T_0 f + P_f - R_f, & f(0) = f_0 \\[2ex] \dfrac{\partial g}{\partial t} = T_0 g + P_g - R_g, & g(0) = g_0 \end{cases}$$

where P_f, P_g, R_f, R_g are the bilinear operators defined previously in (1.3) and (1.4), with domains $D(T_0)^2 \subset \mathscr{X}$ where $f_0, g_0 \in D(T_0) \cap X^+ \cap D(\tilde{T}_0)$. If such a system admits a solution, that is a pair of functions f and g with values in X which are strongly continuously differentiable with respect to

$t \in \mathbb{R}$, belong to $D(T_0)$, and satisfy the differential equations and initial conditions, then the pair f and $h := f + g$ satisfy

$$\begin{cases} \dfrac{\partial f}{\partial t} = T_0 f + h \displaystyle\int_V f\,dw - f \int_V h\,dw, & f(0) = f_0 \\ \dfrac{\partial h}{\partial t} = \tilde{T}_0 h, \qquad h(0) = h_0 = f_0 + g_0 \end{cases}$$

not only in $\mathscr{X} = X^2$, but in $X \times Y$. The results quoted above assure that $h(x, v, t) = h_0(x - vt, v)$ and consequently that the Cauchy problem (PTP) has a unique strongly continuously differentiable solution defined for any $t \in \mathbb{R}$, $f_0, g_0 \in D(T_0) \cap D(\tilde{T}_0) \cap X^+$ given by

$$f(t) = U(t, 0)\, f_0$$
$$g(t) = \tilde{U}(t, 0)\, g_0$$

Since $\|f(t)\|_X = \|f_0\|_X$, from (2.2) $\|h(t)\|_X = \|h_0\|_X$ and the norm of L^1 is positively additive it follows that $\|g(t)\|_X = \|g_0\|_X$. Note that $U(t, 0)$ and $\tilde{U}(t, 0)$ depend on $h_0 = f_0 + g_0$.

In order to define the time dependent wave operators we consider as the unperturbed dynamics that driven by the family $U_0(t, s) := e^{(t-s)\,T_0}$, $t, s \in \mathbb{R}$, and as the perturbed dynamics that driven by the family $U(t, s)$. The time dependent wave operators are defined for $s \in \mathbb{R}$, $h_0 \in D(T_0) \cap D(\tilde{T}_0) \cap X^+$, $f \in X$ by

$$W_+(s)\, f := \lim_{t \to \infty} U_0(s, t)\, U(t, s)\, f$$

$$W_-(s)\, f := \lim_{t \to -\infty} U(s, t)\, U_0(t, s)\, f$$

provided the limits exist with respect to the norm of the space X. We note that the function $t \mapsto U(t, s)\, f$ behaves like the function $t \mapsto U_0(t, s)\, W_+(s)\, f$ for $t \to +\infty$, and the function $t \mapsto U_0(t, s)\, f$ behaves like the function $t \mapsto U(t, s)\, W_-(s)\, f$ for $t \to -\infty$. We are able to determine sufficient conditions on h_0 in order that the wave operators exist.

Theorem 2.1. Let $h_0 \in D(T_0) \cap D(\tilde{T}_0) \cap X^+$ and $H(x, t)$ be defined as in (2.3). Then for all $f \in X$, and all $s \geqslant 0$, the wave operator $W_+(s)\, f$ exists if

$$\alpha_+ := \int_s^{+\infty} \sup_{x \in \mathbb{R}^3} H(x, r)\, dr < \infty \tag{2.5}$$

and $W_{-}(s)\, f$ exists if

$$\alpha_{-} := \int_{-\infty}^{s} \sup_{x \in \mathbb{R}^3} H(x, r)\, dr < \infty \tag{2.6}$$

By using the positivity of the operators $U_0(s, r)$, $-A(r)$, and $K(r)$ for $s, r \in \mathbb{R}$, of $U(r, s)$ for $r \geqslant s$, of the equality $\| U_0(s, t)\, U(t, s)\, f \|_X = \| f \|_X$, for all $f \geqslant 0$, $t \geqslant s$, and of careful uses of Duhamel's formulae, it is proved in [Bus1] that

$$\alpha_{+} := \sup_{t \geqslant s} \sup_{(y, w) \in \mathbb{R}^3 \times V} \int_{s}^{t} H(y + w(r - t), r)\, dr < \infty$$

and

$$\alpha_{-} := \sup_{(y, w) \in \mathbb{R}^3 \times V} \int_{-\infty}^{s} H(y - w(s - r), r)\, dr < \infty$$

are the sufficient conditions for the existence of the wave operators $W_{+}(s)$ and $W_{-}(s)$. Hence (2.5) and (2.6) are the obvious consequence of these results.

3. CHARACTERIZATION OF THE EVOLUTION GROUP

Denote by $E := C_0(\mathbb{R}, X)$ the space of continuous X-valued functions vanishing at $\pm\infty$, with the supremum norm. By a slight modification all the results of this section can be carried over to $E := L^p(\mathbb{R}, X)$ (see [Eva]).

The right translations $\{\mathbf{T}_0(t)\}_{t \in \mathbb{R}}$, acting on E are defined by

$$\mathbf{T}_0(t)\, \psi(s) := \psi(s - t) \qquad \text{for} \quad s, t \in \mathbb{R} \tag{3.1}$$

and are a C_0-group on E. This group is well understood and its generator is $-\mathbf{d}$;

$$\mathbf{d}\psi := \psi' \qquad \text{with} \quad D(\mathbf{d}) := \{\psi \in E \cap C^1(\mathbb{R}, X) \mid \psi' \in E\}$$

Now, let us define a bounded multiplication operator on E as follows: For each $\phi \in C_0(\mathbb{R})$, the operator $\mathscr{M}(\phi)$ on E defined by

$$[\mathscr{M}(\phi)\, \psi](t) = \phi(t)\, \psi(t) \qquad \text{for all} \quad \psi \in E$$

is called *bounded scalar multiplication operator* on E. Let $\{A(t)\}_{t\in\mathbb{R}}$ be a family of strongly continuous bounded operators on X such that $\|A(\cdot)\|_\infty := \sup_{t\in\mathbb{R}} \|A(t)\| < \infty$. To such a family we can associate an element $\mathscr{A}$ of $\mathscr{L}(E)$ by

$$[\mathscr{A}\psi](t) = A(t)\,\psi(t) \qquad \text{for all} \quad \psi \in E$$

such that $\mathscr{A}$ commutes with $\mathscr{M}(\phi)$, for all $\phi \in C_0(\mathbb{R})$. Such an operator is called *bounded multiplication operator* on E. We have $\|\mathscr{A}\| = \|A(\cdot)\|_\infty$ and we refer to [Eva] for the characterization of these operators. It is not hard to verify that if the operators $A(t)$ are invertible in $\mathscr{L}(X)$, then $\mathscr{A}$ is also invertible and $\mathscr{A}^{-1}$, given by $[\mathscr{A}^{-1}\psi](t) = A(t)^{-1}\,\psi(t)$, for any $\psi \in E$. This consideration allows us to define an invertible bounded multiplication operator $\mathscr{U}$ on E that acts in the following way for any $t \in \mathbb{R}$ and any $\psi \in E$

$$[\mathscr{U}\psi](t) \equiv U(t, 0)\,\psi(t)$$

and write

$$e^{t\mathscr{G}} = \mathscr{U}\mathbf{T}_0(t)\,\mathscr{U}^{-1} \tag{3.2}$$

Now we are in a position to specify the particular properties of the evolution group $e^{t\mathscr{G}}$ when the corresponding $U(t, s)$ is the propagator for charged particles transport problem (NTP) defined in the previous sections.

Theorem 3.1. Under the assumptions of Section 2, $e^{t\mathscr{G}}$ is a strongly continuous group of isometries in E. This evolution group $e^{t\mathscr{G}}$ is similar to the group of translations $\mathbf{T}_0(t)$.

Proof. The group property $e^{(t+s)\mathscr{G}} = e^{t\mathscr{G}}e^{s\mathscr{G}}$ is equivalent to the Chapman–Kolmogorov equation (P3). Denote by $C_c(\mathbb{R}, X)$ the space of continuous functions $\psi\colon \mathbb{R} \mapsto X$ with compact support and norm $\|\psi\| = \sup_{s\in\mathbb{R}} \|\psi(s)\|_X$. For $\psi \in C_c(\mathbb{R}, X)$ we obtain $e^{t\mathscr{G}}\psi \in C_c(\mathbb{R}, X)$, since

$$\|e^{t\mathscr{G}}\psi - \psi\| = \sup_{s\in\mathbb{R}} \|U(s, s-t)\,\psi(s-t) - \psi(s)\|_X$$

the properties (P1) and (P4) of $U(t, s)$ imply that $\lim_{t\to 0} \|e^{t\mathscr{G}}\psi - \psi\| = 0$ and the strong continuity follows from the density of $C_c(\mathbb{R}, X)$ in E. By virtue of (2.4) and the fact that the right translation (3.1) is isometric, it follows that $e^{t\mathscr{G}}$ is also isometric in E. Finally, the similarity with $\mathbf{T}_0(t)$ follows from (3.2). ▮

The operators $A(t)$, $K(t)$ and consequently $B(t)=A(t)+K(t)$ are uniformly bounded in X. For $T(t)=T_0+A(t)+K(t)$ with $D(T(t))=D(T_0)$ and for all $f\in D(T(t))$ we have

$$\frac{\partial}{\partial t}U(t,s)\,f=T(t)\,U(t,s)\,f \qquad \text{and} \qquad \frac{\partial}{\partial s}U(t,s)\,f=-U(t,s)\,T(s)\,f$$

Let us denote by $\mathscr{B}$ the bounded multiplication operator on E corresponding to the family $\{B(t)\}_{t\in\mathbb{R}}$ on X and by $\mathscr{T}$ the unbounded multiplication operator corresponding to the family $\{(T(t),D(T_0))\}_{t\in\mathbb{R}}$ of unbounded operators on X. The operator $\mathscr{T}$ with domain

$$D(\mathscr{T}):=\left\{\psi\in E\,\middle|\,\lim_{t\to 0}\frac{1}{t}\,[\,U(\cdot,\cdot-t)-I\,]\,\psi(\cdot)\in E\right\} \tag{3.3}$$

is defined by

$$[\mathscr{T}\psi](\cdot):=\lim_{t\to 0}\frac{1}{t}\,[\,U(\cdot,\cdot-t)-I\,]\,\psi(\cdot) \tag{3.4}$$

In the same manner we can define the multiplication operator $\mathscr{T}_0$ on the space E by replacing $U(t,s)$ with $U_0(t,s)$ in (3.3) and (3.4). The following Theorem characterizes $\mathscr{G}$ the generator of the evolution group $e^{t\mathscr{G}}$.

Theorem 3.2. Let $h_0\in D(T_0)\cap D(\tilde{T}_0)\cap X^+$. With the notation above $\mathbf{D}:=D(\mathbf{d})\cap D(\mathscr{T})$ is a core for $\mathscr{G}$ and for any $\psi\in\mathbf{D}$ we have

$$\mathscr{G}\psi=\mathscr{T}\psi-\psi'=\mathscr{T}_0\psi+\mathscr{B}\psi \tag{3.5}$$

Proof. For $\psi\in E$ and $t>0$, let us denote by

$$\delta(t)=\frac{1}{t}\,(e^{t\mathscr{G}}\psi-\psi)=\delta_1(t)+\delta_2(t) \tag{3.6}$$

where

$$\delta_1(t)=\frac{1}{t}\,U(\cdot,\cdot-t)(\psi(\cdot-t)-\psi(\cdot))$$

and

$$\delta_2(t)=\frac{1}{t}\,(U(\cdot,\cdot-t)\,\psi(\cdot)-\psi(\cdot))$$

For $\psi \in D(\mathbf{d}) \cap D(\mathscr{T})$, $t \to 0$, $\delta_1(t)$ converges to $-\psi'$ and $\delta_2(t)$ converges to $\mathscr{T}\psi$ in E. Hence $\delta(t)$ converges to $\mathscr{G}\psi$ which implies that $\psi \in D(\mathscr{G})$. Conversely if $\psi \in D(\mathbf{d}) \cap D(\mathscr{G})$, then the same argument yields $\psi \in D(\mathscr{T})$. This means that

$$\mathbf{D} = D(\mathbf{d}) \cap D(\mathscr{T}) = D(\mathbf{d}) \cap D(\mathscr{G})$$

Since $-\mathbf{d}$ and $\mathscr{G}$ are the generators of the C_0-semigroups $\mathbf{T}_0(t)$ and $e^{t\mathscr{G}}$, $D(\mathbf{d})$, $D(\mathscr{G})$ and consequently $\mathbf{D}$ are dense in E.

In order to prove that $\mathbf{D}$ is a core for $\mathscr{G}$, it suffices to show that it is invariant under the action of the semigroup $e^{t\mathscr{G}}$ (see [Dav, Theorem 1.9]). Since for $\psi \in D(\mathscr{G})$ one has $e^{t\mathscr{G}}\psi \in D(\mathscr{G})$, it suffices to show that $e^{t\mathscr{G}}\psi \in D(\mathbf{d})$ for $\psi \in \mathbf{D}$. For this we perform an idea of [Nic, Theorem 2.8] and we write formally for $\psi \in D(\mathscr{G})$,

$$\begin{aligned}(e^{t\mathscr{G}}\psi)'(s) &= \frac{\partial}{\partial s} U(s, s-t)\, \psi(s-t) \\ &= \frac{\partial}{\partial \tau} U(\tau, s-t)\Big|_{\tau=s} \psi(s-t) + \frac{\partial}{\partial \tau} U(s, \tau)\Big|_{\tau=s-t} \psi(s-t) \\ &\quad + U(s, s-t)\, \psi'(s-t) \end{aligned} \tag{3.7}$$

Since

$$\begin{aligned}\left(\frac{\partial}{\partial t} e^{t\mathscr{G}}\psi\right)(s) &= \frac{\partial}{\partial t} U(s, s-t)\, \psi(s-t) \\ &= -\frac{\partial}{\partial \tau} U(s, \tau)\Big|_{\tau=s-t} \psi(s-t) - U(s, s-t)\, \psi'(s-t)\end{aligned}$$

the last two terms of (3.7) are well defined and the first term is

$$\frac{\partial}{\partial \tau} U(\tau, s-t)\Big|_{\tau=s} \psi(s-t) = T(s)\, U(s, s-t)\, \psi(s-t)$$

In order to prove that

$$\lim_{|s| \to \infty} \|T(s)\, U(s, s-t)\, \psi(s-t)\|_X = 0$$

we will decompose $T(s)$ as $T(s) = T_0 + A(s) + K(s)$, will use the uniform boundedness of $A(s)$ and $K(t)$ (see [Bus1, Lemma 1]), isometry of $U(s, s-t)$ for $t > 0$ and the fact that

$$\|T_0 U(s, s-t)\, \psi(s-t)\|_X = \|U(s, s-t)\, T_0 \psi(s-t)\|_X = \|T_0 \psi(s-t)\|_X$$

goes to zero as $|s| \to \infty$, for all $\psi \in C_0(\mathbb{R}, D(T_0))$ which is dense in E. Finally (3.5) follows from (3.6) by taking $t \to 0$. ∎

As a consequence of this Theorem we can consider the operator $\mathscr{G}$ as the closure of $\mathscr{G}$ defined by (3.5) on **D**.

In [Bus1, Theorem 2], it is shown that $T_0 + A(t)$ generates the following propagator

$$[U_a(t, s)\, f](x, v) = f(x - (t-s)\, v, v) \exp\left\{ -\int_s^t H(x - (t-r)\, v, r)\, dr \right\} \quad (3.8)$$

and that $T_0 + A(t) + K(t)$ generates a propagator $U(t, s)$ which satisfies the Duhamel's formula

$$U(t, s) = U_a(t, s) + \int_s^t U_a(t, r)\, K(r)\, U(r, s)\, dr$$

Hence, for $\psi \in E$,

$$\begin{aligned} &U(s, s-t)\, \psi(s-t) \\ &\quad = U_a(s, s-t)\, \psi(s-t) + \int_{s-t}^{s} U_a(s, r)\, K(r)\, U(r, s-t)\, \psi(s-t)\, dr \end{aligned} \quad (3.9)$$

Now, if we denote by $\{e^{t\mathscr{G}_a}\}_{t \in \mathbb{R}}$ the evolution group given by

$$[e^{t\mathscr{G}_a} \psi](\cdot) = U_a(\cdot, \cdot - t)\, \psi(\cdot - t), \qquad \text{for all} \quad \psi \in E$$

by putting $\tau = r - s + t$ in the integral of (3.9), we obtain the Duhamel's formula for $e^{t\mathscr{G}}$; namely

$$e^{t\mathscr{G}} = e^{t\mathscr{G}_a} + \int_0^t e^{(t-\tau)\,\mathscr{G}_a} \mathscr{K} e^{\tau\mathscr{G}}\, d\tau \quad (3.10)$$

where $\mathscr{K}$ is the bounded multiplication operator corresponding to the family $\{K(t)\}_{t \in \mathbb{R}}$. In the same way we can write

$$e^{t\mathscr{G}} = e^{t\mathscr{G}_0} + \int_0^t e^{(t-\tau)\,\mathscr{G}} \mathscr{B} e^{\tau\mathscr{G}_0}\, d\tau \quad (3.11)$$

Since $U_0(t, s)$, $U_a(t, s)$ are positive for all $t, s \in \mathbb{R}$ and $U(t, s)$ is positive for $t \geqslant s$, we obtain the positivity of $e^{t\mathscr{G}_0}$, $e^{t\mathscr{G}_a}$ for all $t \in \mathbb{R}$ and positivity of $e^{t\mathscr{G}}$ for $t \geqslant 0$ on the Banach lattice E. Furthermore, since $\{K(t)\}_{t \in \mathbb{R}}$ is a family of positive operators in X, for any $\psi \geqslant 0$ in E the function $t \mapsto \int_0^t e^{(t-\tau)\mathscr{G}_a} \mathscr{K} e^{\tau \mathscr{G}} \psi \, d\tau$ is positively increasing in E, consequently

$$e^{t\mathscr{G}} \geqslant e^{t\mathscr{G}_a} \qquad \text{for all} \quad t \geqslant 0 \tag{3.12}$$

Lemma 3.3. Let $h_0 \in X^+$, and $H(x, t) = \int_V h_0(x - tv, v)\, dv$, then

$$\|e^{t\mathscr{G}_a}\psi\|_E \leqslant \begin{cases} \|\psi\|_E & \text{if} \quad t \geqslant 0 \\ \sup\limits_{s,x,v} \exp\left\{\int_0^{-t} H(x+\sigma v, s+\sigma)\, d\sigma\right\} \|\psi\|_E & \text{if} \quad t \leqslant 0 \end{cases} \tag{3.13}$$

Proof. For $t \geqslant 0$, $\|e^{t\mathscr{G}_a}\psi\|_E \leqslant \|\psi\|_E$ is direct consequence of (3.8), and for $t \leqslant 0$,

$$\begin{aligned} \|[e^{t\mathscr{G}_a}\psi](s)\|_X &= \|U_a(s, s-t)\, \psi(s-t)\|_X \\ &\leqslant \|\psi(s-t)\|_X \sup_{x,v} \exp\left\{\int_0^{-t} H(x+\sigma v, s+\sigma)\, d\sigma\right\} \end{aligned}$$

This implies (3.13). ▮

4. STATIONARY SCATTERING THEORY

In the context of the charged particles transport equation, suppose that one is given two bounded evolution groups defined by (1.9) and (1.12). Since $U_0(t, s)$ is isometric in X for all $t, s \in \mathbb{R}$, thus $e^{t\mathscr{G}_0}$ is also isometric in E and $U(t, s)$ being isometric for $t \geqslant s$, thus $e^{t\mathscr{G}}$ is also isometric in E for $t \geqslant 0$. Hence one can view the existence of two wave operators

$$\mathscr{W}_+(\mathscr{G}_0, \mathscr{G}) \equiv s - \lim_{t \to +\infty} e^{-t\mathscr{G}_0} e^{t\mathscr{G}} \tag{4.1$_+$}$$

$$\mathscr{W}_-(\mathscr{G}, \mathscr{G}_0) \equiv s - \lim_{t \to -\infty} e^{-t\mathscr{G}} e^{t\mathscr{G}_0} \tag{4.1$_-$}$$

and the related scattering operator $\mathscr{S}$ defined by (1.13) in E.

There are two ways to prove an existence theorem for (4.1). The first way is to use Theorem 2.1 together with the fact that for $\psi \in E$,

$$[e^{-t\mathscr{G}_0} e^{t\mathscr{G}} \psi](\cdot) = U_0(\cdot, \cdot + t)\; U(\cdot + t, .)\; \psi(\cdot) \tag{4.2$_+$}$$

and

$$[e^{-t\mathcal{G}} e^{t\mathcal{G}_0} \psi](\cdot) = U(\cdot, \cdot + t)\, U_0(\cdot + t, \cdot)\, \psi(\cdot) \tag{4.2$_-$}$$

by taking $t \to \pm\infty$ in $(4.2_\pm)$. These considerations show that the wave operators $\mathcal{W}_\pm$ are the multiplication operators by $W_\pm(\cdot)$ and the scattering operator $\mathcal{S}$ is therefore the multiplication operator by $S(\cdot)$. If we denote by $S = S(0)$ the usual scattering operator which is referred to initial time $t = 0$, then $\mathcal{U}_0 \mathcal{S} \mathcal{U}_0^{-1}$ is multiplication by S (see [Eva, Theorem 2.2]).

The second way is to apply directly the smooth perturbation theory for the existence of $\mathcal{W}_\pm$. The uniform sufficient conditions for the existence of $W_\pm(\cdot)$ in X imply the existence of $\mathcal{W}_\pm$ in E. This method has the advantage of not using the regularity condition on the initial datum h_0.

Theorem 4.1. Let $h_0 \in X_+$, $H(x, t) = \int_V h_0(x - tv, v)\, dv$ and

$$\alpha_- = \sup_{(x, v, s) \in \mathbb{R}^3 \times V \times \mathbb{R}} \int_{-\infty}^{0} H(x + tv, s)\, dt < \infty \tag{4.3}$$

Then the wave operator $\mathcal{W}_-(\mathcal{G}, \mathcal{G}_0)$ exists in $\mathcal{L}(E)$.

Proof. Suppose that for any ψ in some dense subspace of E we have

$$\int_{-\infty}^{0} \|\mathcal{B} e^{t\mathcal{G}_0} \psi\|_E\, dt < \infty \tag{4.4}$$

By applying the Duhamel's formula (3.11) for $t < 0$, we obtain

$$e^{-t\mathcal{G}} e^{t\mathcal{G}_0} \psi = \psi + \int_t^0 e^{-s\mathcal{G}} \mathcal{B} e^{s\mathcal{G}_0} \psi\, ds \tag{4.5}$$

By virtue of Cook's lemma, the existence of the limit (4.1_-) follows from (4.4), while $\|e^{t\mathcal{G}}\|_E = 1$ for $t \geqslant 0$.

In order to prove (4.4), we remark that

$$\begin{aligned} \|\mathcal{B} e^{t\mathcal{G}_0} \psi\|_E &= \sup_{s \in \mathbb{R}} \|B(s)\, U_0(s, s - t)\, \psi(s - t)\|_X \\ &= \sup_{s \in \mathbb{R}} \int_{\mathbb{R}^3 \times V} |(A(s) + K(s))\, \psi(s - t, x - tv, v)|\, dx\, dv \end{aligned}$$

Since for $h_0 \geqslant 0$ we have

$$\int_{\mathbb{R}^3 \times V} |K(s)\, \psi(s-t, x-tv, v)|\, dx\, dv$$

$$\leqslant \int_{\mathbb{R}^3 \times V} h_0(x-sv, v) \int_V |\psi(s-t, x-tv', v')|\, dv'\, dx\, dv$$

$$= \int_{\mathbb{R}^3 \times V} H(x, s)\, |\psi(s-t, x-tv', v')|\, dx\, dv'$$

$$= \int_{\mathbb{R}^3 \times V} |A(s)\, \psi(s-t, x-tv, v)|\, dx\, dv$$

the following estimate

$$\int_{-\infty}^{0} \|\mathscr{B} e^{t\mathscr{G}_0} \psi\|_E\, dt$$

$$\leqslant 2 \sup_{s \in \mathbb{R}} \int_{-\infty}^{0} \int_{\mathbb{R}^3 \times V} H(x, s)\, |\psi(s-t, x-tv, v)|\, dx\, dv$$

$$\leqslant 2 \left(\sup_{(x, v, s) \in \mathbb{R}^3 \times V \times \mathbb{R}} \int_{-\infty}^{0} H(x+tv, s)\, dt \right) \|\psi\|_E \tag{4.6}$$

together with (4.3) imply the Theorem. ∎

For the existence of $\mathscr{W}_+(\mathscr{G}_0, \mathscr{G})$ we need the following Lemma.

Lemma 4.2. Let $h_0 \in X_+$, $H(x, t) = \int_V h_0(x-tv, v)\, dv$ and

$$\alpha_+ = \sup_{(x, v, s) \in \mathbb{R}^3 \times V \times \mathbb{R}} \int_{0}^{\infty} H(x+tv, s)\, dt < \infty \tag{4.7}$$

Then the wave operator $\mathscr{W}_+(\mathscr{G}_0, \mathscr{G})$ exists iff $\mathscr{W}_+(\mathscr{G}_a, \mathscr{G})$ exists in $\mathscr{L}(E)$.

Proof. Suppose that the wave operator $\mathscr{W}_+(\mathscr{G}_0, \mathscr{G}_a)$ exists. Since

$$e^{-t\mathscr{G}_0} e^{t\mathscr{G}} \psi = e^{-t\mathscr{G}_0} e^{t\mathscr{G}_a} e^{-t\mathscr{G}_a} e^{t\mathscr{G}} \psi$$

the wave operator $\mathscr{W}_+(\mathscr{G}_0, \mathscr{G})$ exists iff $\mathscr{W}_+(\mathscr{G}_a, \mathscr{G})$ exists in $\mathscr{L}(E)$. For the existence of $\mathscr{W}_+(\mathscr{G}_0, \mathscr{G}_a)$ we write

$$[e^{-t\mathscr{G}_0} e^{t\mathscr{G}_a}]\, \psi(s) = U_0(s, s+t)\, U_a(s+t, s)\, \psi(s)$$

$$= \exp\left\{ \int_0^t h(x+\tau v, \tau+s)\, d\tau \right\} \psi(s)$$

for all $\psi \in E$. Thus, Lebesgue's dominated convergence theorem, together with (4.7), shows the existence of $\mathscr{W}_+(\mathscr{G}_0, \mathscr{G}_a)$. ∎

Remark 4.3. As the existence of $\mathscr{W}_+(\mathscr{G}_0, \mathscr{G}_a)$ is implied by (4.7), it follows readily that this condition implies also the existence of $\mathscr{W}_-(\mathscr{G}_a, \mathscr{G}_0)$. Similarly the existence of $\mathscr{W}_-(\mathscr{G}_0, \mathscr{G}_a)$ and $\mathscr{W}_+(\mathscr{G}_a, \mathscr{G}_0)$ follows from (4.3).

Theorem 4.4. Under the hypothesis of Lemma 4.2, the wave operator $\mathscr{W}_+(\mathscr{G}_0, \mathscr{G})$ exists in $\mathscr{L}(E)$.

Proof. By virtue of Lemma 4.2, it suffices to show that $\mathscr{W}_+(\mathscr{G}_a, \mathscr{G})$ exist. The action of the semigroup $e^{-t\mathscr{G}_a}$ on $e^{t\mathscr{G}}$ given by (3.11) implies that

$$e^{-t\mathscr{G}_a} e^{t\mathscr{G}} \psi = \psi + \int_0^t e^{-s\mathscr{G}_a} \mathscr{K} e^{s\mathscr{G}} \psi \, ds, \qquad \text{for all} \quad \psi \in E$$

The Lemma 3.3 yields

$$\int_0^\infty \| e^{-r\mathscr{G}_a} \mathscr{K} e^{r\mathscr{G}} \psi \|_E \, dr \leqslant \int_0^\infty \exp\left(\int_0^r H(x + tv, t + r) \, dt \right) \| \mathscr{K} e^{r\mathscr{G}} \psi \|_E \, dr$$

$$\leqslant \alpha_+ \int_0^\infty \| \mathscr{K} e^{r\mathscr{G}} \psi \|_E \, dr$$

for all $\psi \in E$. On the other hand

$$\int_0^\infty \| \mathscr{K} e^{r\mathscr{G}} \psi \|_E \, dr \leqslant \sup_{s \in \mathbb{R}} \int_0^\infty \| K(s) \, U(s, s - r) \, \psi(s - r) \|_X \, ds$$

$$= \sup_{s \in \mathbb{R}} \int_{-\infty}^s \| K(t + r) \, U(t + r, t) \, \psi(t) \|_X \, dt$$

$$\leqslant \beta \, \| U(\cdot + r, \cdot) \, \psi(\cdot) \|_E \leqslant \beta \, \| \psi \|_E$$

where $\beta = e^{\alpha_+} - 1$. Here the last inequality is shown in [Bus1, p. 205]. Hence the existence of the wave operator $\mathscr{W}_+(\mathscr{G}_0, \mathscr{G})$ follows from Cook's lemma. ∎

5. SIMILARITY BETWEEN PERTURBED AND UNPERTURBED OPERATORS

In the last section we have analysed the existence of the wave operators $\mathscr{W}_+(\mathscr{G}_0, \mathscr{G})$ and $\mathscr{W}_-(\mathscr{G}, \mathscr{G}_0)$ given by $(4.1)_\pm$. In the same way one can prove the existence of four wave operators $\mathcal{W}_{\pm}(\mathcal{G}_0, \mathcal{G})$ and $\mathcal{W}_{\pm}(\mathcal{G}, \mathcal{G}_0)$ given by

$$\mathcal{W}_{\pm}(\mathcal{G}_0, \mathcal{G}) \equiv s - \lim_{t \to \pm\infty} e^{-t\mathcal{G}_0} e^{t\mathcal{G}} \qquad (5.1_{\pm})$$

$$\mathcal{W}_{\pm}(\mathcal{G}, \mathcal{G}_0) \equiv s - \lim_{t \to \pm\infty} e^{-t\mathcal{G}} e^{t\mathcal{G}_0} \qquad (5.2_{\pm})$$

It is a standard result that the existence of each wave operator implies the correspondent intertwining identity in the following list

(a)

$$e^{t\mathcal{G}} \mathcal{W}_{\pm}(\mathcal{G}, \mathcal{G}_0) f = \mathcal{W}_{\pm}(\mathcal{G}, \mathcal{G}_0) e^{t\mathcal{G}_0} f; \qquad f \in E$$

(b)

$$\mathcal{G} \mathcal{W}_{\pm}(\mathcal{G}, \mathcal{G}_0) f = \mathcal{W}_{\pm}(\mathcal{G}, \mathcal{G}_0) \mathcal{G}_0 f; \qquad f \in D(\mathcal{G}_0)$$

(c)

$$e^{t\mathcal{G}_0} \mathcal{W}_{\pm}(\mathcal{G}_0, \mathcal{G}) f = \mathcal{W}_{\pm}(\mathcal{G}_0, \mathcal{G}) e^{t\mathcal{G}} f; \qquad f \in E$$

(d)

$$\mathcal{G}_0 \mathcal{W}_{\pm}(\mathcal{G}_0, \mathcal{G}) f = \mathcal{W}_{\pm}(\mathcal{G}_0, \mathcal{G}) \mathcal{G} f; \qquad f \in D(\mathcal{G})$$

Furthermore we have

$$\mathcal{W}_{\pm}(\mathcal{G}, \mathcal{G}_0) \mathcal{W}_{\pm}(\mathcal{G}_0, \mathcal{G}) f = \mathcal{W}_{\pm}(\mathcal{G}_0, \mathcal{G}) \mathcal{W}_{\pm}(\mathcal{G}, \mathcal{G}_0) f = f; \qquad f \in E$$

These relations imply that

$$\mathcal{W}_{\pm}(\mathcal{G}, \mathcal{G}_0)^{-1} = \mathcal{W}_{\pm}(\mathcal{G}_0, \mathcal{G}) \qquad \text{and} \qquad \mathcal{W}_{\pm}(\mathcal{G}_0, \mathcal{G})^{-1} = \mathcal{W}_{\pm}(\mathcal{G}, \mathcal{G}_0)$$

Hence the existence of the pair $(\mathcal{W}_{-}(\mathcal{G}, \mathcal{G}_0), \mathcal{W}_{-}(\mathcal{G}_0, \mathcal{G}))$ or $(\mathcal{W}_{+}(\mathcal{G}, \mathcal{G}_0), \mathcal{W}_{+}(\mathcal{G}_0, \mathcal{G}))$ implies the similarity of two operators $\mathcal{G}$ and $\mathcal{G}_0$ and the correspondent C_0-groups in the following sense

$$\mathcal{W}_{\pm}(\mathcal{G}_0, \mathcal{G}) \mathcal{G} \mathcal{W}_{\pm}(\mathcal{G}, \mathcal{G}_0) f = \mathcal{G}_0 f, \qquad f \in D(\mathcal{G}_0)$$

$$\mathcal{W}_{\pm}(\mathcal{G}, \mathcal{G}_0) \mathcal{G}_0 \mathcal{W}_{\pm}(\mathcal{G}_0, \mathcal{G}) f = \mathcal{G} f, \qquad f \in D(\mathcal{G})$$

$$\mathcal{W}_{\pm}(\mathcal{G}_0, \mathcal{G}) e^{t\mathcal{G}} \mathcal{W}_{\pm}(\mathcal{G}, \mathcal{G}_0) f = e^{t\mathcal{G}_0} f, \qquad f \in E$$

$$\mathcal{W}_{\pm}(\mathcal{G}, \mathcal{G}_0) e^{t\mathcal{G}_0} \mathcal{W}_{\pm}(\mathcal{G}_0, \mathcal{G}) f = e^{t\mathcal{G}} f, \qquad f \in E$$

The following theorem ensures the existence of the wave operators $(5.1_{\pm})$ and $(5.1_{\pm})$.

Theorem 5.1. Let α_- and α_+ be the real numbers defined in (4.3) and (4.6). If one of the conditions

$$\alpha_- < \tfrac{1}{2} \tag{5.3}$$

or

$$\alpha_+ < \tfrac{1}{2} \tag{5.4}$$

are satisfied, then the four wave operators $\mathscr{W}_{\pm}(\mathscr{G}_0, \mathscr{G})$, $\mathscr{W}_{\pm}(\mathscr{G}, \mathscr{G}_0)$ exist.

Remark that for similarity between $\mathscr{G}$ and $\mathscr{G}_0$, we do not need to prove the existence of all the wave operators. The existence of the pair $(\mathscr{W}_+(\mathscr{G}_0, \mathscr{G}), \mathscr{W}_+(\mathscr{G}, \mathscr{G}_0))$ or $(\mathscr{W}_-(\mathscr{G}_0, \mathscr{G}), \mathscr{W}_-(\mathscr{G}, \mathscr{G}_0))$ is sufficient for proving this similarity. But, as we will see, the existence of $\mathscr{W}_+(\mathscr{G}, \mathscr{G}_0))$ is conditioned by (5.3) and this condition infers automatically the existence of the three other wave operators. A similar situation occurs for the existence of $\mathscr{W}_-(\mathscr{G}_0, \mathscr{G})$.

The proof of the above Theorem can be performed through a sequence of Lemmas.

Lemma 5.2. The following assertions are equivalent

(a)

$$\int_{-\infty}^{0} \|\mathscr{B}e^{t\mathscr{G}_0}f\|_E \, dt \leqslant \gamma \, \|f\|_E, \qquad f \in E \tag{5.5}$$

(b)

$$\int_{0}^{\infty} \|\mathscr{B}e^{t\mathscr{G}_0}f\|_E \, dt \leqslant \gamma \, \|f\|_E, \qquad f \in E \tag{5.6}$$

Proof. Take $r > 0$ and $s = t - r$ in (5.5), then

$$\begin{aligned}\int_0^r \|\mathscr{B}e^{t\mathscr{G}_0}f\|_E \, dt &= \int_{-r}^{0} \|\mathscr{B}e^{(s+r)\mathscr{G}_0}f\|_E \, ds \\ &\leqslant \gamma \, \|e^{r\mathscr{G}_0}f\|_E \\ &\leqslant \gamma \, \|f\|_E\end{aligned}$$

Since r is an arbitrary number, we get (5.6). The proof of the converse is similar. ∎

Lemma 5.3. Suppose that in one of the relations (a) or (b) of Lemma 5.2 we have $\gamma < 1$. Then the C_0-group $\{e^{t\mathcal{G}}\}_{t \in \mathbb{R}}$ is uniformly bounded on $\mathbb{R}$.

Proof. For this we use Duhamel's formula which asserts that

$$e^{t\mathcal{G}} = e^{t\mathcal{G}_0} + \int_0^t e^{(t-s)\mathcal{G}} \mathcal{B} e^{s\mathcal{G}_0}\, ds, \qquad t \in \mathbb{R}$$

Let $\tau > 0$. For $t \in [0, \tau]$ and $f \in X$, from (5.5) we have

$$\begin{aligned} \|e^{-t\mathcal{G}} f\|_E &\leqslant \|e^{-t\mathcal{G}_0} f\|_E + \sup_{s \in [0, \tau]} \|e^{-s\mathcal{G}}\|_{\mathcal{L}(E)} \int_{-\tau}^0 \|\mathcal{B} e^{s\mathcal{G}_0} f\|\, ds \\ &\leqslant \|f\|_E + \gamma \sup_{s \in [0, \tau]} \|e^{-s\mathcal{G}}\|_{\mathcal{L}(E)} \|f\|_E \end{aligned}$$

and therefore,

$$\sup_{t \in [0, \tau]} \|e^{-t\mathcal{G}}\|_{\mathcal{L}(E)} \leqslant \frac{1}{1-\gamma}$$

Since τ is choosen arbitrary, we conclude the boundedness of $\{e^{t\mathcal{G}}\}_{t \in \mathbb{R}}$ on $\mathbb{R}_-$. The same argument gives the boundedness of $\{e^{t\mathcal{G}}\}_{t \in \mathbb{R}}$ on $\mathbb{R}_+$ by using (5.6). ∎

Lemma 5.4. Under the assumption of Lemma 5.3 we have one of the following equivalent assertions.

(a)

$$\int_0^{\infty} \|\mathcal{B} e^{t\mathcal{G}} f\|_E\, dt < \infty, \qquad f \in E \tag{5.7}$$

(b)

$$\int_{-\infty}^{0} \|\mathcal{B} e^{t\mathcal{G}} f\|_E\, dt < \infty, \qquad f \in E \tag{5.8}$$

Proof. By Lemma 5.2, (5.5) and (5.6) are both satisfied. Hence $e^{t\mathcal{G}}$ is uniformly bounded on $\mathbb{R}$. This implies the equivalence between (a) and (b).

In fact assume (a) and apply the principle of uniform boundedness, then there exists a constant $C \geqslant 0$, such that

$$\int_0^\infty \|\mathscr{B}e^{t\mathscr{G}}f\|_E\,dt \leqslant C\,\|f\|_E, \qquad f \in E$$

Let $r > 0$ and $s = t + r$, then

$$\begin{aligned}\int_{-r}^0 \|\mathscr{B}e^{t\mathscr{G}}f\|_E\,dt &= \int_0^r \|\mathscr{B}e^{(s-r)\,\mathscr{G}}f\|_E\,ds\\ &\leqslant C\,\|e^{-r\mathscr{G}}f\|_E\\ &\leqslant CM\,\|f\|_E\end{aligned}$$

r being arbitrary, one gets (b). For the converse we may argue in the similar way.

Thus, for the proof of this Lemma we have only to show one of the statement of this Lemma. Let us prove (5.8).

For this, we present the C_0-group $e^{t\mathscr{G}}$ by its Dyson–Phillips expansion given by

$$e^{t\mathscr{G}} = \sum_{n=0}^{\infty} \mathscr{T}_n(t) \tag{5.9}$$

where $\mathscr{T}_0(t) = e^{t\mathscr{G}_0}$, and

$$\mathscr{T}_n(t) = \int_0^t e^{(t-s)\,\mathscr{G}_0}\mathscr{B}\mathscr{T}_{n-1}(s)\,ds, \qquad (n \geqslant 1) \tag{5.10}$$

For any $f \in E$,

$$\int_{-\infty}^0 \|\mathscr{B}e^{t\mathscr{G}}f\|_E\,dt \leqslant \int_{-\infty}^0 \left\|\mathscr{B}\sum_{n=0}^{\infty}\mathscr{T}_n(t)\,f\right\|_E dt$$

hence it is enough to show that

$$\sum_{n=0}^{\infty}\int_{-\infty}^0 \|\mathscr{B}\mathscr{T}_n(t)\,f\|_E\,dt \leqslant C\,\|f\|_E \tag{5.11}$$

where, $C = (1-\gamma)^{-1}$. We shall prove by induction that

$$\int_{-\infty}^0 \|\mathscr{B}\mathscr{T}_n(t)\,f\|_E\,dt \leqslant \gamma^{n+1}\,\|f\|$$

which is given in the assumptions for $n = 0$. Suppose

$$\int_{-\infty}^{0} \|\mathscr{B}\mathscr{T}_{n-1}(t)\, f\|_E \, dt \leqslant \gamma^n \, \|f\|_E \tag{5.12}$$

Then,

$$\begin{aligned}
\int_{-\infty}^{0} \|\mathscr{B}\mathscr{T}_n(t)\, f\|_E \, dt &= \int_{-\infty}^{0} \left\| \mathscr{B} \left\{ -\int_t^0 e^{(t-s)\,\mathscr{G}_0} \mathscr{B}\mathscr{T}_{n-1}(s)\, f\, ds \right\} \right\|_E dt \\
&\leqslant \int_{-\infty}^{0} \int_{-\infty}^{0} \|\mathscr{B} e^{t\mathscr{G}_0} [\, e^{-s\mathscr{G}_0} \mathscr{B}\mathscr{T}_{n-1}(s)\, f\,]\|_E \, dt\, ds \\
&\leqslant \gamma \int_{-\infty}^{0} \|e^{-s\mathscr{G}_0} \mathscr{B}\mathscr{T}_{n-1}(s)\, f\|_E \, ds \qquad \text{(by (5.5))} \\
&\leqslant \gamma \int_{-\infty}^{0} \|\mathscr{B}\mathscr{T}_{n-1}(s)\, f\|_E \, ds \\
&\leqslant \gamma^{n+1} \, \|f\|_E \qquad \text{(by (5.12))}
\end{aligned}$$

This implies (5.11). ▌

Proof of the Theorem 5.1. Suppose that one of the conditions (5.3) or (5.4), say (5.3), is satisfied then according to Theorem 4.1 the wave operator $\mathscr{W}_-(\mathscr{G}, \mathscr{G}_0)$ exists and (4.6) shows (5.5) with $\gamma < 1$. As the Duhamel's formula asserts (4.5), one can write also that

$$e^{-t\mathscr{G}} e^{t\mathscr{G}_0} \psi = \psi - \int_0^t e^{-s\mathscr{G}} \mathscr{B} e^{s\mathscr{G}_0} \psi \, ds \tag{5.13}$$

and

$$e^{-t\mathscr{G}_0} e^{t\mathscr{G}} \psi = \psi + \int_0^t e^{-s\mathscr{G}_0} \mathscr{B} e^{s\mathscr{G}} \psi \, ds \tag{5.14}$$

The Lemma 5.2 asserts that (5.5) implies (5.6) and since $e^{t\mathscr{G}}$ is uniformly bounded on $\mathbb{R}_-$, by letting $t \to \infty$ in (5.13) one gets the existence of $\mathscr{W}_+(\mathscr{G}, \mathscr{G}_0)$. For the the existence of $\mathscr{W}_\pm(\mathscr{G}_0, \mathscr{G})$, one takes $t \to \pm\infty$ in (5.14) and uses Lemma 5.4. ▌

All the above results concerning the existence of the wave operators lead us naturally to the notion of $\mathscr{G}_0$-*smoothness.*

Definition 5.5. Let $\mathscr{G}_0$ be the generator of a C_0-group $e^{t\mathscr{G}_0}$ on a Banach space E. A linear operator $\mathscr{B}$ is called $\mathscr{G}_0$-*smooth* with constant $\alpha > 0$, if

$$\int_{-\infty}^{\infty} \|\mathscr{B}e^{t\mathscr{G}_0}f\|_E \, dt \leqslant \alpha \, \|f\|_E \tag{5.15}$$

holds for a dense set of vectors f in E (and hence for all f in E).

Corollary 5.6. Let $\alpha = \alpha_- + \alpha_+$. If $\mathscr{B}$ is $\mathscr{G}_0$-smooth with constant $\alpha < 1$. Then the wave operators $\mathscr{W}_\pm(\mathscr{G}, \mathscr{G}_0)$ and $\mathscr{W}_\pm(\mathscr{G}_0, \mathscr{G})$ exist.

Proof. One of the conditions (5.3) or (5.4) is necessarily satisfied. Since the contrary leads to contradicting with $\alpha < 1$. Thus the Theorem 5.1 implies the result. ∎

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